Imbalance, Cutwidth, and the Structure of Optimal Orderings

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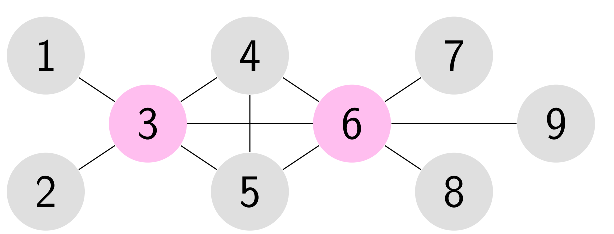
Imbalance Minimization [1]

Let $G = (V, E)$ be a graph and σ an ordering of V. For $v \in V$, let $\text{pred}_\sigma(v) = |\sigma_{<v} \cap N(v)|$ and $\text{succ}_\sigma(v) = |\sigma_{>v} \cap N(v)|$. The *imbalance* of v w.r.t. σ, denoted $\phi_\sigma(v)$, is $|\text{succ}_\sigma(v) - \text{pred}_\sigma(v)|$. The *imbalance* of σ is $im(\sigma) = \sum_{v \in \sigma} \phi_\sigma(v)$. $im(G)$, the *imbalance* of G, is the minimum of $im(\sigma)$ over all orderings σ of V.



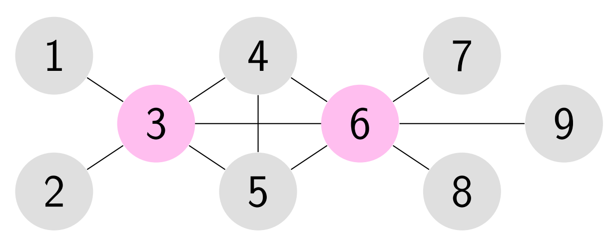
$\sigma = \langle 1, 2, 3, 4, 5, 6, 7, 8, 9 \rangle$

v	1	2	3	4	5	6	7	8	9
$\phi_\sigma(v)$	1	1	1	1	1	0	1	1	1

$im(\sigma) = 8$

Cutwidth Minimization

Let $G = (V, E)$ be a graph and σ an ordering of V. The *cutwidth* after v w.r.t. σ, denoted $c_\sigma(v)$, is $c_\sigma(v) = |\{(x, y) \in E \mid x \leq_\sigma v \text{ and } v <_\sigma y\}|$. The *cutwidth* of σ is $cw(\sigma) = \max_{v \in \sigma}\{c_\sigma(v)\}$. $cw(G)$, the *cutwidth* of G is the minimum of $cw(\sigma)$ over all orderings σ of V.



$\sigma = \langle 1, 2, 3, 4, 5, 6, 7, 8, 9 \rangle$

v	1	2	3	4	5	6	7	8	9
$c_\sigma(v)$	0	1	2	3	4	3	3	2	1

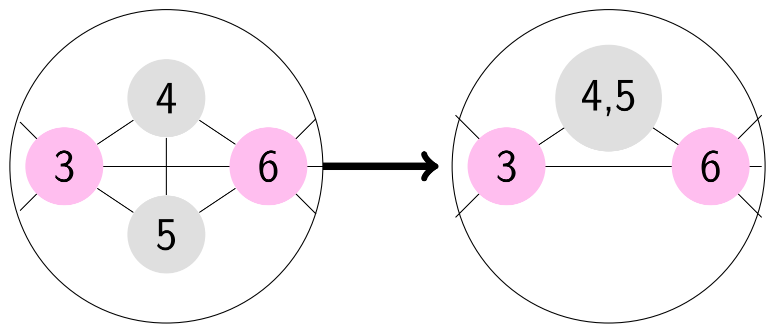
$cw(\sigma) = 4$

Twin-Cover

Two vertices u and v are *twins* if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Two twins u and v are *true twins* if $N[u] = N[v]$. A *twin cover* of G is a set $T \subseteq V$ s.t. for every edge $(u, v) \in E$, either $\{u, v\} \cap T \neq \emptyset$ or u and v are twins. The minimum size of a twin cover is denoted $tc(G)$.

Theorem

$\forall\ G$, $\exists\ \sigma$ s.t. $im(\sigma) = im(G)$, and all true twins are grouped together. $\exists$ FPT algorithm for $im(G)$ parameterized by $tc(G)$.



Theorem

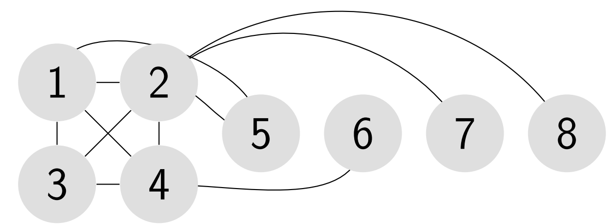
$\forall\ G$, $\exists\ \sigma$ s.t. $cw(\sigma) = cw(G)$, and all true twins are grouped together. $\exists$ FPT algorithm for $cw(G)$ parameterized by $tc(G)$.

Split Graphs

A *split* graph is a graph that can be partitioned into a clique and an independent set.

Theorem

Computing the imbalance of a split graph is NP-complete.



Theorem ([2])

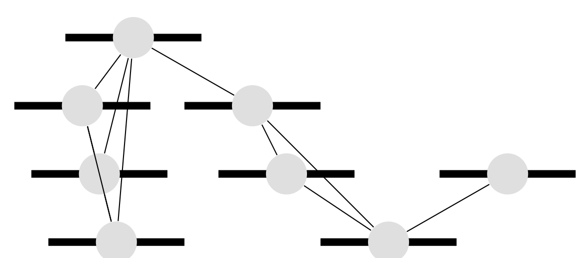
Computing the cutwidth of a split graph is NP-complete.

Proper Interval (Bi)Graphs

An *interval* graph is one in which each vertex v may be identified with an interval I_v of the real line, s.t. $(u, v) \in E$ iff $I_u \cap I_v \neq \emptyset$. An interval graph is *proper* if no interval contains another.

Theorem

Computing the imbalance of a proper-interval graph can be done in $O(n)$.



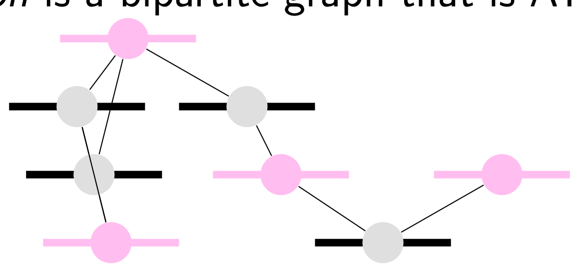
Theorem ([2])

Computing the cutwidth of a proper-interval graph can be done in $O(n)$.

An *asteroidal triple* (AT) is a triple of independent vertices x, y, z such that between every pair of vertices, there is a path that does not intersect the closed neighbourhood of the third. A *proper interval bigraph* is a bipartite graph that is AT-free and contains no induced cycle of size greater than 4.

Theorem

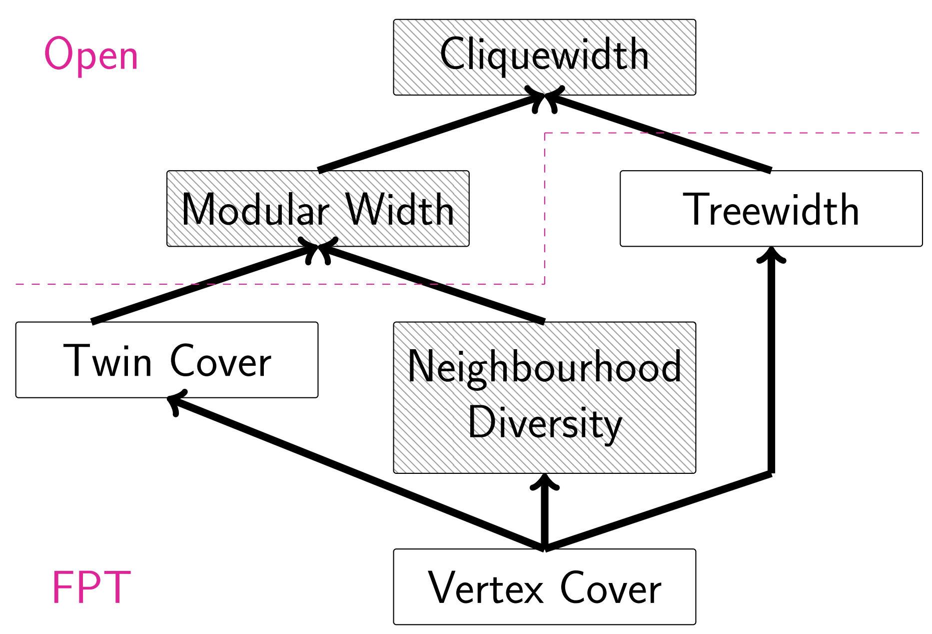
Computing the imbalance of a proper-interval bigraph can be done in $O(n)$.



Theorem ([3])

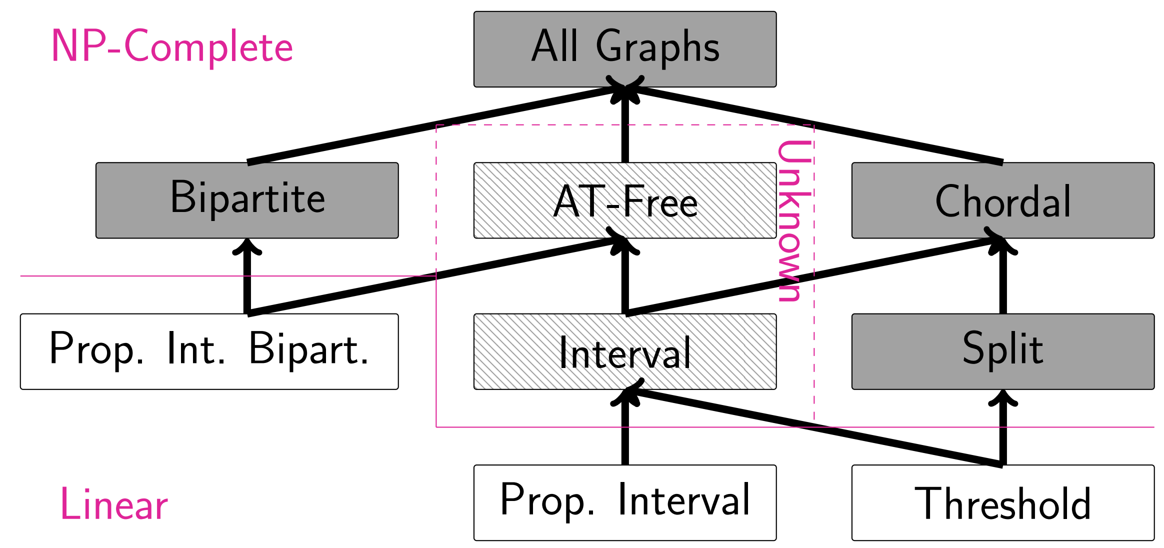
Computing the cutwidth of a proper-interval bigraph can be done in $O(n)$.

Parameters



An arrow from class A to class B indicates that class A is generalized by class B.

Graph Classes



An arrow from class A to class B indicates that class A is contained within class B.

References

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[3] P. Heggernes, P. van't Hof, D. Lokshtanov, and J. Nederlof.
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