End-Vertices of AT-free Bigraphs

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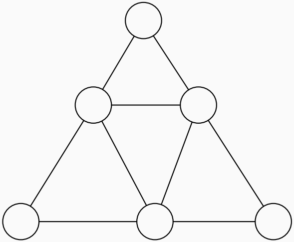
Plan

Graph Searches

A graph *search* is a systematic way of visiting each vertex in a graph.



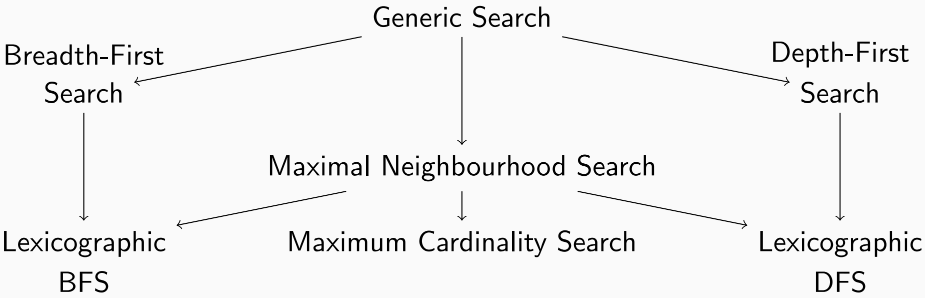
a graph G

$$\sigma = v_1, \ldots, v_n$$

an *ordering* of G

Applications to e.g., connectivity (BFS, DFS) [CLRS09], recognition of graph classes, computation of graph parameters, and detection of certain graph structures (LBFS) [Cor04], computing maximal cardinality matchings on cocomparability graphs (LDFS) [MNN18]

Relationship Between Searches



End-Vertices

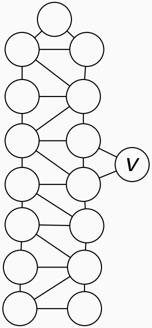
End-Vertex Problem for S:
Instance: A graph $G = (V, E)$, and a vertex t.
Question: Is there an ordering σ generated by S of V such that $\sigma(n) = t$?

Understanding End-Vertices is Helpful I

- End-vertices of LBFS on chordal graphs are simplicial [RTL76]

Led to recognition algorithms for this graph class via a *perfect elimination ordering*: $\forall v$ and $N(v)$ that are after v are a complete graph

...but not every simplicial vertex is an LBFS end-vertex:



Understanding End-Vertices is Helpful II

- End-vertices of LBFS of an interval graph are precisely the simplicial and admissible vertices [COS09]

Led to recognition algorithms for this graph class: six executions of LBFS, tie-breaking based on the previous execution when possible.

- End-vertices of LDFS on cocomparability graphs starts a hamiltonian path, if one exists [CDH13]

Determining If A Vertex is An End Vertex Is Sometimes Easy

For generic search - using a set rather than a queue - v is end-vertex iff:

- v is not a cut-vertex.

Can test this quickly (DFS).

For LBFS on interval graphs v is end-vertex iff:

- v is simplicial and admissible.

End-Vertex Complexity

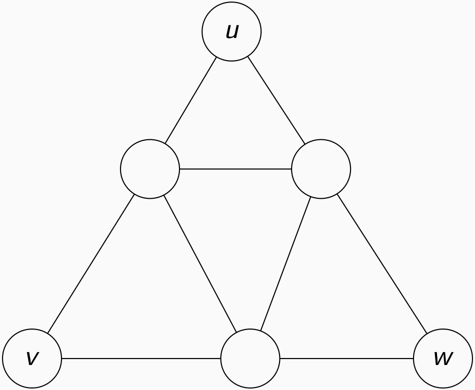
Class	BFS	DFS	LBFS	LDFS	MNS	MCS
All Graphs	NPC	NPC	NPC	NPC	NPC	NPC [BDK+19]
Weakly Chordal	NPC [CHM14]	NPC	NPC [CKL10] [CRWW19]	NPC [CHM14]	NPC [BDK+19]	NPC [CRWW19]
Chordal	?	NPC	?	Linear [CRWW19]	Linear [BDK+19]	Poly. [CRWW19]
Interval	Linear [CRWW19]	Linear [BDK+19]	Linear [CKL10]	Linear	Linear [BDK+19]	Poly.
Unit Interval	Linear	Linear [BDK+19]	Linear [CKL10]	Linear [BDK+19]	Linear [BDK+19]	Linear [BDK+19]
Split	Poly. [CHM14]	NPC [CHM14]	Linear [BDK+19]	Linear [BDK+19]	Linear [BDK+19]	Linear [BDK+19]
Bigraph	NPC [CHM14]	NPC [Gor15]	NPC [GH17]	?	?	?
AT-Free Bigraph	**Poly.**	**Linear**	Poly. [GH17] $\rightarrow$**Linear**	?	**Poly.**	?

Complexity results for the end-vertex problem. Bold results are new.

Asteroidal Triples

Definition (Asteroial Triple)

An *asteroidal triple* is a triple of vertices (u, v, w) such that between any two of them, there is a path that does not intersect the closed neighbourhood of the third.



AT-Free Bigraphs

Definition (AT-Free Bigraphs)

A graph G is an *AT-free Bigraph* if it is bipartite and asteroidal-triple free.

These are exactly the bipartite permutation graphs and the proper interval bipartite graphs [HH04]

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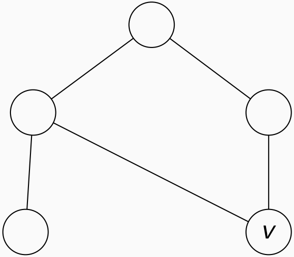
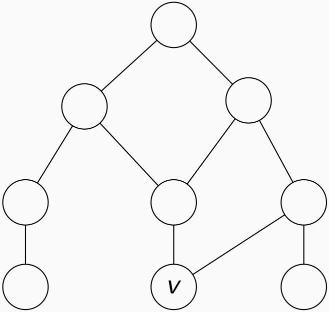
BFS on AT-Free Bigraphs I



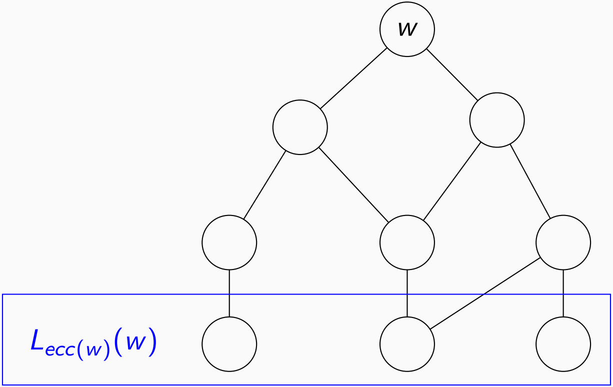
v is not a LBFS end-vertex, but it is a BFS end-vertex.



v is not a BFS end-vertex.

Eccentricity

For a vertex v, the *eccentricity* of v is $\max_{u \in V} d(u, v)$.

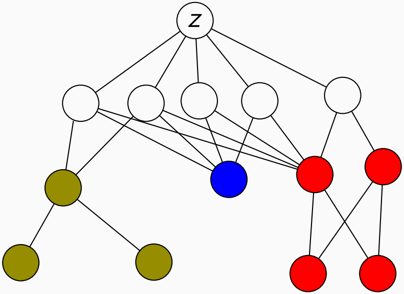


If v is an end-vertex for BFS, it must be the case that there is a w such that $d(w, v) = ecc(w)$.

Layers of AT-Free Bigraphs

Proposition ([GH17])

Let G be a connected AT-free bigraph and z be a vertex of G. If $ecc(z) \geq 3$, then $G - N[z]$ has at most two deep components. □



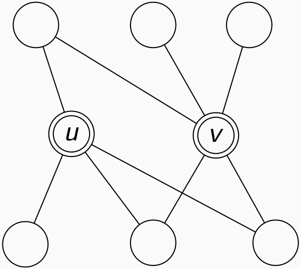
Red, olive are deep; blue is not.

deep components have > 1 vertex

Layers II

Proposition ([GH17])

Let G be an AT-free bigraph and z be a vertex of G. Suppose that C is a connected component of $G - N[z]$ and that $u, v \in L_\ell(z)$ are two vertices in C. The neighbourhoods of u and v are comparable on the levels above and below them.



Level $\ell - 1$ of a component

Level ℓ of a component

Level $\ell + 1$ of a component

BFS End-Vertices of AT-Free Bigraphs

Theorem

Let $G = (V, E)$ be an AT-free bigraph, and $v \in V$ where $3 \leq height(v) < \infty$. Let $w \in V$ be such that $d(w, v) = ecc(w)$. The vertex v is a BFS end-vertex if and only if, either

- *$G - N[w]$ has a single deep component; or*
- *$G - N[w]$ has two deep components C_1 and C_2, where without loss of generality $v \in C_1$, and there is a set $X \subseteq N(w) \cap N(C_2)$ such that for each $u \in Y(v, w)$, $d(X, v) \geq d(X, u)$ and $d(X, u') = ecc(w) - 1$ for all $u' \in Z(v, w)$.*

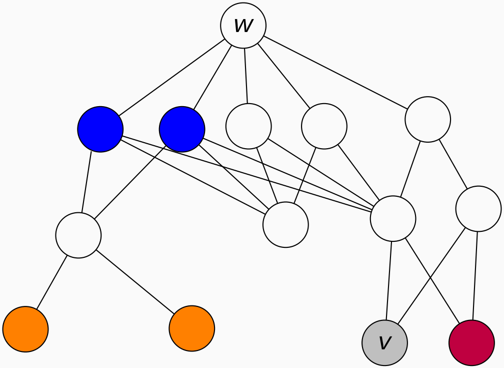
$G - N[w]$ has a single deep component follows from "layers" proposition.

Theorem

The BFS end-vertex problem is in P for AT-free bigraphs.

BFS End-Vertices of AT-Free Bigraphs

There is a set $X \subseteq N(w) \cap N(C_2)$ such that for each $u \in Y(v, w)$, $d(X, v) \geq d(X, u)$ and $d(X, u') = \text{ecc}(w) - 1$ for all $u' \in Z(v, w)$.



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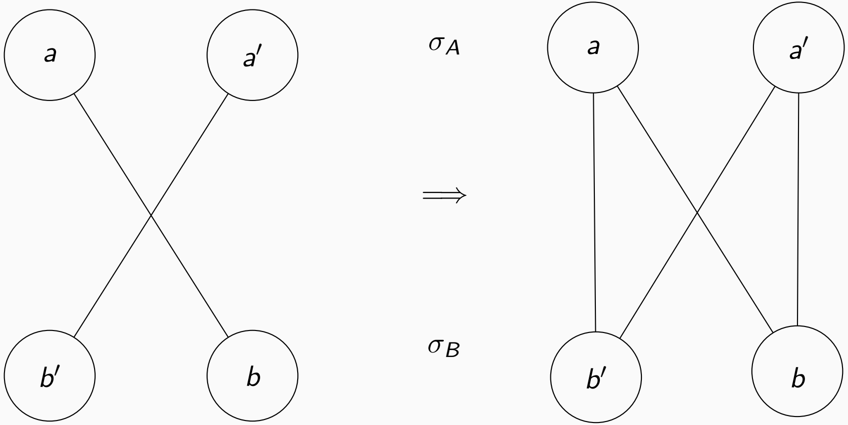
Strong Ordering

A *strong ordering* (σ_A, σ_B) of a bipartite graph $G = (A, B, E)$ consists of an ordering σ_A of A and an ordering σ_B of B such that for all $ab, a'b' \in E$, where $a, a' \in A$ and $b, b' \in B$, $a <_{\sigma_A} a'$ and $b' <_{\sigma_B} b$ implies that $ab' \in E$ and $a'b \in E$.



AT-Free Bigraphs Have Strong Orders with the Adjacency Property

Theorem ([SBS87])

The following statements are equivalent for any bigraph $G = (A, B, E)$.

1. *G has a strong ordering.*
2. *G is a AT-free.* □

Moreover, there exists a strong ordering of G which has the adjacency property.

Helper Theorems

Theorem ([KLM15])

Let G be a connected graph, and let t be a vertex of G. Then t is a DFS end-vertex of G if and only if there is $X \subseteq V(G)$ such that $N_G[t] \subseteq X$ and $G[X]$ has a Hamiltonian path with endpoint t. □

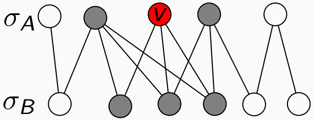
Theorem ([BK85])

If G is an AT-free bigraph, then G has a Hamiltonian cycle if and only if for each $1 \leq i \leq |A|$, if there is a $1 \leq j \leq |B|$ such that $a_i b_j \in E$ and $a_{i+1} b_{j+1} \in E$, then $a_i b_{j+1} \in E$, and $a_{i+1} b_j \in E$.

DFS End-Vertices on AT-Free Bigraphs

Corollary

Let (σ_a, σ_b) be a strong ordering of an AT-free bigraph. Without loss of generality, assume $v \in A$. Let $N(v) = \langle b_\alpha, \ldots, b_\beta \rangle$ in σ_B. v is a DFS-end vertex if and only if there there is a set $A' \subseteq \langle a_i, \ldots, a_j \rangle$ such that $a_i \leq_{\sigma_A} a$, $a \leq_{\sigma_A} a_j$, and $G' = G[A' \cup N(v)]$ is an induced subgraph of G where $|A'| = |N(v)|$ and G' has a Hamiltonian path ending at v.



Theorem

The DFS end-vertex problem has a linear-time solution for AT-free bigraphs.

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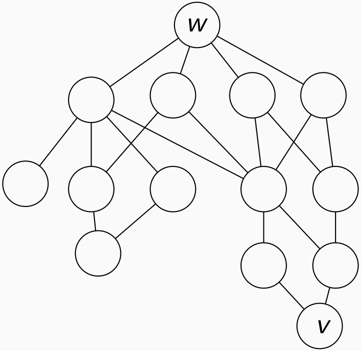
Characterization

Theorem ([GH17])

Let G be a connected AT-free bigraph and v be a vertex of G. Then v is the end-vertex of an LBFS if and only if there exists a vertex w such that, for every eccentric vertex u of w, $N(u) \supseteq N(v)$. □

Understanding the Characterization I

"there exists a vertex w such that, for every eccentric vertex u of w, $N(u) \supseteq N(v)$"



...But some vertices are LBFS ends even with two "eccentric deep components"!

LBFS on AT-Free Bigraphs

Theorem

Let G be a connected AT-free bigraph of $diam(G) = k$. Suppose that v with $ecc(v) = k - 1$ is the end-vertex of LBFS of G. If w is the first vertex of $L_r(v)$ where $r = \lceil \frac{ecc(v)+3}{2} \rceil$ in an LBFS ordering of G that begins at v, then for every eccentric vertex u of w, $N(u) \supseteq N(v)$.

Linear Time Solution

Algorithm 1: Linear Time LBFS End-Vertex for AT-Free Bigraphs

Input: An AT-free graph G with vertex v.

Output: A vertex w such that $N(u) \supseteq N(v)$ for every eccentric vertex u of w in which case v is the end-vertex of G, or else v is not the end-vertex of LBFS of G.

1 Run LBFS begining at v to find an LBFS ordering τ. Let $w_1 = \tau(n)$ and w_2 be the first vertex of $L_r(v)$ in τ where $r = \lceil \frac{ecc(v)+3}{2} \rceil$

2 Run LBFS begining at w_1 to find all eccentric vertices of w_1. If $N(u) \supseteq N(v)$ for all eccentric vertices u of w_1, v is the end-vertex of LBFS of G.

3 Run LBFS begining at w_2 to find all eccentric vertices of w_2. If $N(u) \supseteq N(v)$ for all eccentric vertices u of w_2, v is the end-vertex of G; otherwise v is not the end-vertex of LBFS of G.

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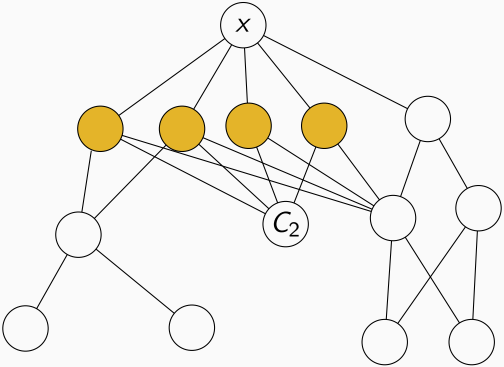
Substars

Definition ([BBBS10])

Let $G = (V, E)$ be a graph and $x \in V$. Let $C_1, \ldots, C_k$ be the connected components of $G(V - N[x])$. The *substars* of x are the elements of $N_G(C_i))$ for each i. □



MNS End-Vertices

Theorem ([BBBS10])

Let G be a connected **chordal** *and let x be a vertex of G. Then the following statements are equivalent:*

1. *the substars of x are totally ordered by inclusion;*
2. *x is an MNS-end vertex.*

Theorem

Let G be a connected **AT-free bigraph** *and let x be a vertex of G. Then the following statements are equivalent:*

1. *the substars of x are totally ordered by inclusion;*
2. *x is an MNS-end vertex.*

Theorem

The MNS end-vertex problem is in P for AT-free bigraphs.

Plan

Conclusion

Class	BFS	DFS	LBFS	LDFS	MNS	MCS
Bigraph	NPC [CHM14]	NPC [Gor15]	NPC [GH17]	?	?	?
AT-Free Bigraph	**Poly.**	**Linear**	Poly. [GH17]$\rightarrow$**Linear**	?	**Poly.**	?

- Future Work
  - MCS, LDFS end-vertex complexity on AT-free bigraphs?
  - LDFS, MNS, MCS end-vertex complexity on general bigraphs?

Thank you.

Questions? Comments?

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