Imbalance on Threshold Graphs and Bipartite Permutation Graphs

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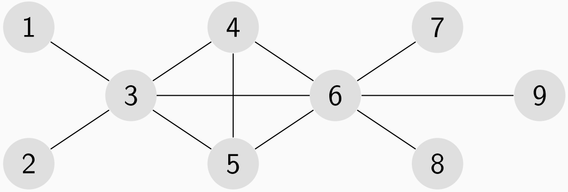
Plan

Our Problem: Imbalance

- A *linear layout* problem: given a graph G, embed the vertices on a path of length $|V(G)|$ and minimize some function $f()$.
    - In our case: f represents the sum of the difference of each vertex's neighbourhood to its left and right in the embedding.
- First introduced by Biedl et al. [BCG$^+$05]; various applications in graph drawing [Kan96, KH97, PT98, Woo03, Woo04].
- NP-complete for split graphs and on bipartite graphs ($\Delta \leq 6$); it has a linear solution on trees and proper interval graphs [BCG$^+$05, **?**].

Imbalance Visualized



$$\sigma = \langle 1, 2, 3, 4, 5, 6, 7, 8, 9 \rangle$$

$$N(4) = \{3, 5, 6\} = \langle 1, 2, \boxed{3,}\ 4, \boxed{5, 6,}\ 7, 8, 9 \rangle$$

$$\phi_\sigma(4) = \big| |\{3\}| - |\{5, 6\}| \big| = 1$$

v	1	2	3	4	5	6	7	8	9
$\phi_\sigma(v)$	1	1	1	1	1	0	1	1	1

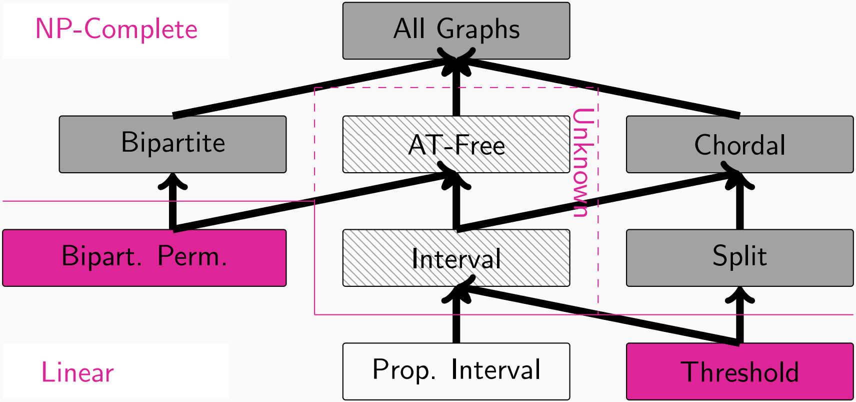
$$im(\sigma) = \sum_{v \in V} \phi_\sigma(v) = 8$$

Imbalance Minimization [BCG+05]

Definition

Let $G = (V, E)$ be a graph and σ an ordering of V. For $v \in V$, let $pred_\sigma(v) = |\sigma_{<v} \cap N(v)|$ and $succ_\sigma(v) = |\sigma_{>v} \cap N(v)|$. The imbalance of v w.r.t. σ, denoted $\phi_\sigma(v)$, is $|succ_\sigma(v) - pred_\sigma(v)|$. The imbalance of σ is $im(\sigma) = \sum_{v \in \sigma} \phi_\sigma(v)$. $im(G)$, the imbalance of G, is the minimum of $im(\sigma)$ **over all orderings** *σ of V.*

Graph Class Results



An arrow from class A to class B indicates that class A is contained within class B. Pink classes are in this work.

Plan

Bipartite Permutation Graphs

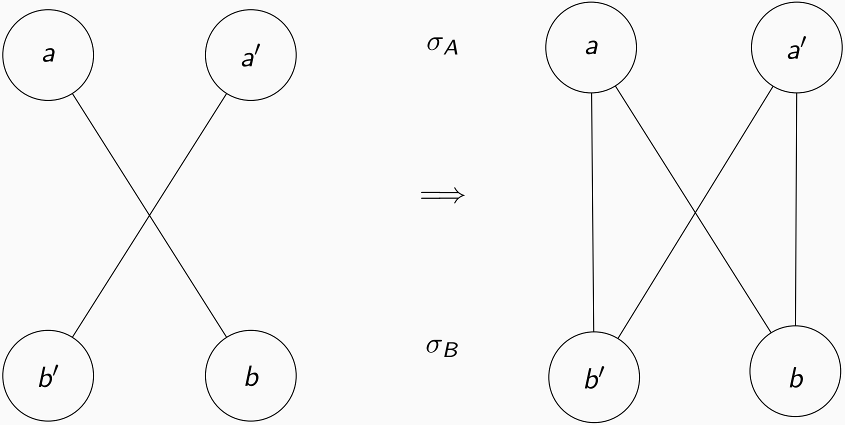
A graph is a *permutation* graph if it is the intersection graph of lines whose end points are on two parallel lines. A graph is a bipartite permutation graph if it is both a bipartite graph and a permutation graph.

Proper interval bipartite graphs are bipartite permutation graphs [HH04].

Complete bipartite graphs are bipartite permutation graphs.

Strong Ordering

A *strong ordering* (σ_A, σ_B) of a bipartite graph $G = (A, B, E)$ consists of an ordering σ_A of A and an ordering σ_B of B such that for all $ab, a'b' \in E$, where $a, a' \in A$ and $b, b' \in B$, $a <_{\sigma_A} a'$ and $b' <_{\sigma_B} b$ implies that $ab' \in E$ and $a'b \in E$.



Orderings of Bipartite Permutation Graphs

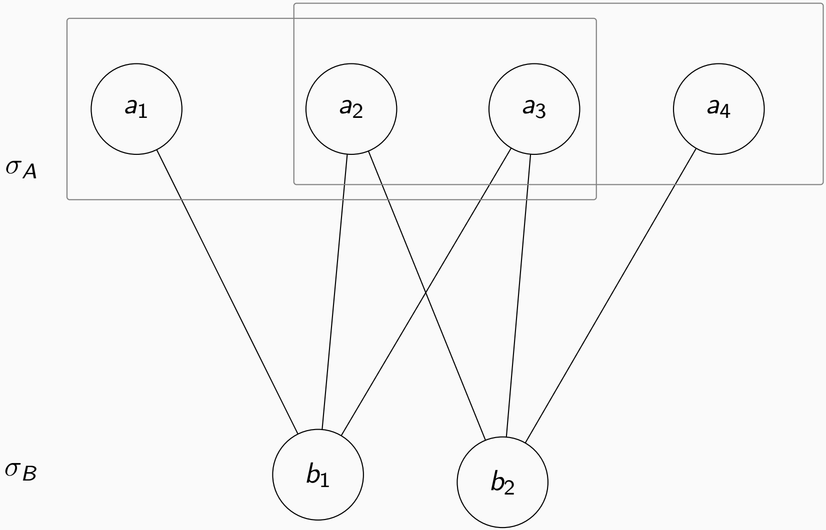
Theorem ([SBS87])

The following statements are equivalent for any bipartite graph $G = (A, B, E)$.

1. *G is a bipartite permutation graph.*
2. *G has a strong ordering.*
3. *There exists an ordering of A which has the adjacency property and the enclosure property.*

A strong ordering of a bipartite permutation graph can be computed in linear time [CHK99].

Enclosure and Adjacency Properties



Strong Orderings of Bipartite Permutation Graphs

Lemma ([SBS87])

Let (σ^A, σ^B) *be a strong ordering of a connected bipartite permutation graph* $G = (A, B, E)$*. Then both* σ^A *and* σ^B *have the adjacency property and the enclosure property.*

Imbalance on General Bipartite Graphs

Theorem ([BCG$^+$05])

Given a bipartite graph $G = (A, B, E)$ and a fixed vertex-ordering σ^A of A, there is a linear time algorithm that finds an ordering of G which is imbalance-minimal with respect to all orderings that agree with σ^A.

Imbalance on Bipartite Permutation Graphs

Theorem

Let (σ^A, σ^B) *be a strong ordering of a bipartite permutation graph* $G = (A, B, E)$*. There is an ordering* σ *of* G *with* $im(\sigma) = im(G)$ *and* $\sigma_A = \sigma^A$*.*

Proof Sketch

- Handle some small highly structured cases (e.g., $diam(G) \leq 2$).
- Induction on the size of the graph:
  - Given a graph $G = (A, B)$ with n vertices, create G' by removing a_1 and G'' by removing b_t.
  - Get optimal orderings which satisfy the properties for G' and G'', identify v with the same $N(v)$ split in both orderings
  - Glue orderings together at v

Imbalance on Bipartite Permutation Graphs

Corollary

If G is a bipartite permutation graph, $im(G)$ can be computed in linear time.

Proof.

A strong ordering of $G = (A, B, E)$ can be obtained in linear time [CHK99]. Applying Theorem 3 using σ^A generates an optimal ordering relative to σ^A in linear time, which is optimal by Theorem 4. $\square$

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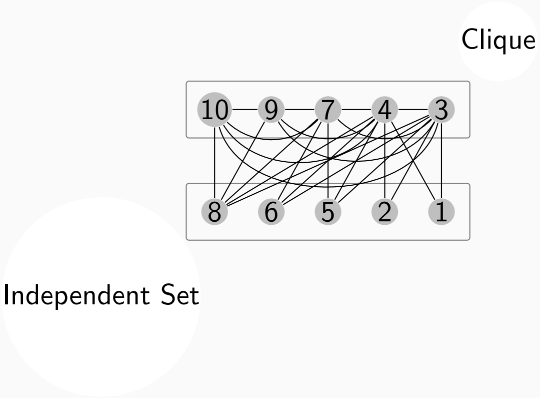
Split Graphs



A split graph, with its split partition indicated.

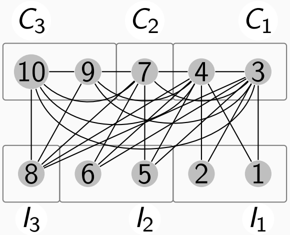
Imbalance is NP-complete on split graphs [**?**].

Threshold Graphs

A graph is a *threshold graph* if and only if it has a split partition (C, I) such that vertices of I (and equivalently the vertices of C) can be ordered by neighbourhood inclusion.

Such a split partition is called a *threshold partition*; computing a threshold partition takes linear time [**?**].

Threshold Partition Visualized



A threshold graph G with levels of its threshold partition indicated.

Key Lemma

Lemma

Let G be a threshold graph on $\ell \geq 3$ levels and let σ be an ordering of G. Suppose that either $|C_1| \geq 2$, or $|C_1| = 1$ and σ is an ordering of G such that I_1 appears as the first $|I_1|$ vertices of σ. Then there is an ordering σ' such that $im(\sigma') \leq im(\sigma)$ and $c_i <_{\sigma'} c_j$ for $c_i \in C_i$ and $c_j \in C_j$ whenever $j > i$.

Proof Sketch

We define a partial order $\lessdot$ on orderings of G, for which each $\lessdot$-minimal ordering has no inversions (c_j, c_i) where $j > i$, $c_j \in C_j$, $c_i \in C_i$, and $c_j <_\sigma c_i$.

We define the ordering $\lessdot$ as follows: if inverted$(\pi) <$ inverted(σ), then $\pi \lessdot \sigma$.

We then show that for any σ in which a pair (c_j, c_i) such that $j > i$, $c_j \in C_j$, $c_i \in C_i$, and $c_j <_\sigma c_i$ appears, there exists π with $\pi \lessdot \sigma$, and $im(\pi) \leq im(\sigma)$.

Proof Sketch II

Given an ordering σ, we will say that a pair (c_j, c_i) is an *inverted pair* if $j > i$ and $c_j <_\sigma c_i$ and $c_j \in C_j$, $c_i \in C_i$; an inverted pair is a *bad pair* if it is also the case that $N(c_i) \cap \sigma_{>c_i} = N(c_j) \cap \sigma_{>c_i}$. For an ordering σ, let inverted(σ) be the number of inverted pairs in σ.

We proceed by contradiction.

Proof Sketch III

Let (c_j, c_i) be the inverted pair that has both c_j and c_i as far right as possible - this actually implies that σ contains a bad pair.

Let (c_j, c_i) be the bad pair that places c_j as far right as possible and minimizes the number of vertices between c_j and c_i in σ.

Establish that the vertices between c_i and c_j are in C_i or I...

Proof Sketch IV

...either we can move one of those vertices, or we have these two constraints:

$$|N(c_j) \cap L| \geq |N(c_j) \cap (M \cup \{c_i\} \cup R)| - |M|. \tag{1}$$

$$|N(c_i) \cap (L \cup \{c_j\} \cup M)| \leq |N(c_i) \cap R| + (|M| + 1), \tag{2}$$

Thus, we have

$$\begin{aligned}
|N(c_i) \cap R| &\geq |N(c_i) \cap (L \cup \{c_j\} \cup M)| - |M| - 1 && \text{by (2)} \\
&= |N(c_i) \cap L| + |\{c_j\}| + |M| - |M| - 1 \\
&= |N(c_i) \cap L| \\
&\geq |N(c_j) \cap L| \\
&\geq |N(c_j) \cap (M \cup \{c_i\} \cup R)| - |M| && \text{by (1)} \\
&\geq |N(c_j) \cap R| + |M| + |\{c_i\}| - |M| > |N(c_j) \cap R|
\end{aligned}$$

which is a contradiction. □

Structured Cases

Lemma

Let G be a threshold graph on $\ell \geq 3$ levels such that $|C_1| = 1$. If $|I_1| \leq |G \setminus (C_1 \cup I_1)|$, then there is an ordering σ such that $im(\sigma) = im(G)$ and I_1 are the first $|I_1|$ vertices of σ.

Lemma

Let G be a threshold graph on $\ell \geq 3$ levels such that $|C_1| = 1$. If $|I_1| > |G \setminus (C_1 \cup I_1)|$, then there is an ordering σ' such that $im(\sigma') = im(G)$ and $c_i <_{\sigma'} c_j$ for $c_i \in C_i$ and $c_j \in C_j$ whenever $j > i$.

Putting It All Together

Lemma

If G be a threshold graph, then there is an ordering σ' such that $im(\sigma') = im(G)$ and $c_i <_{\sigma'} c_j$ for $c_i \in C_i$ and $c_j \in C_j$ whenever $j > i$.

The Final Result

Theorem

Imbalance can be solved in time $O(n)$ for threshold graphs.

Proof.

(Sketch) Let $G'' = (V, E')$, and construct G' by adding each edge $(u, v) \in E$ such that $u, v \in C$ and subdividing it.
Now the graph is bipartite.
By Lemma 9, at least one optimal ordering σ of G is such that $\sigma_C = \tau$.
Apply the algorithm of Theorem 3 to get an optimal ordering σ' of G'. $\square$

Another Structural Result

Corollary

If G be a threshold graph, then there is an ordering σ' such that $im(\sigma') = im(G)$ and $c_i <_{\sigma'} c_j$ for $c_i \in C_i$ and $c_j \in C_j$ whenever $j > i$, and $v_i <_{\sigma'} v_j$ for $v_i \in I_i$ and $v_j \in I_j$ whenever $j > i$.

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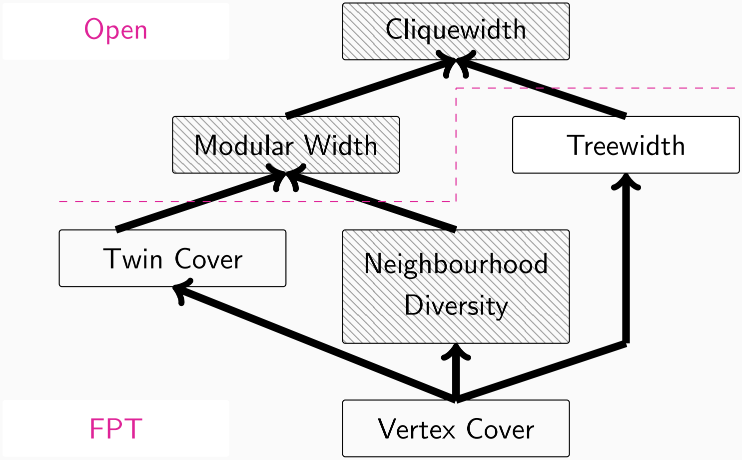
Parameterized Results



An arrow from class A to class B indicates that class A is generalized by class B.

Bounded VC Graphs

Use the approach of Cygan et al. [CLP$^+$14] for Cutwidth.

Theorem

Let G be a graph with vertex cover of size k. There is an algorithm to solve Imbalance in time $O(2^k n^{O(1)})$. Therefore there is a $O(2^{n/2} n^{O(1)})$ time algorithm for Imbalance on bipartite graphs.

Theorem

Imbalance parameterized by the size of the vertex cover does not admit a polynomial kernel, unless $NP \subseteq coNP/poly$.

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Conclusion

- Future Work
  - Imbalance's complexity on cographs? On trivially perfect graphs?
  - Formalization of relationship to cutwidth?

Thank you.

Questions? Comments?

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